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Bifurcations under Weak Noise

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Dynamical systems driven by Gaussian white noise share some general asymptotic properties in the limit of weak noise which are reminiscent of equilibrium thermodynamics. These properties and applications to bifurcating systems are discussed.

KEY WORDS: Weak noise limit; potential conditions; Hamilton–Jacobi equation; bifurcations; exit times; Josephson junctions.

1. THE LIMIT OF WEAK NOISE

In macroscopic physics dynamical systems are, in principle, always perturbed by noise,[1] but in many cases the noise is extremely weak and can therefore safely be neglected, giving rise to a deterministic dynamical system. However, there are special cases, e.g., if the system under study, deterministically, is at or near a bifurcation point, where even weak noise can have a large effect on the long-time dynamics. In such cases the limits of infinite time and vanishing noise intensity do not commute, and it is interesting to study the asymptotic effects of weak but finite noise. In this limit an appealing formal resemblence of the general weak-noise problem to equilibrium thermodynamics emerges, which is useful because it allows one to distinguish those properties which are common to the weak-noise limit and those which are present only in thermodynamic equilibrium. A review of the formalism with applications has recently been given in ref. 2.

A mathematical theory of the weak-noise limit has been developed by Freidlin and Ventsel.[3] Here we only provide the background necessary for the discussion of some applications.[2–6] The dynamical systems under

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consideration are governed by the stochastic differential equations in the sense of Ito,

$$\dot{q}^{\nu} = K^{\nu}(q) + g_i^{\nu}(q)\, \xi^i(t) \tag{1.1}$$

where the $q = \{q^{\nu}; \nu = 1,..., n\}$ are macroscopic variables, and the not explicitly time-dependent functions $K^{\nu}(q)$, $g_i^{\nu}(q)$ $(\nu = 1,..., n,\ i = 1,..., n)$ describe, respectively, the drift and the diffusion of the stochastic process, the diffusion matrix being given by $Q^{\nu\mu}(q) = g_i^{\nu}(q)\, g_i^{\mu}(q)$. The summation convention is implied. $\xi^i(t)$ is Gaussian white noise with intensity η

$$\langle \xi^i(t)\, \xi^j(t') \rangle = \eta \delta^{ij} \delta(t - t') \tag{1.2}$$

The Fokker–Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial q^{\nu}} K^{\nu}(q) P + \frac{\eta}{2} \frac{\partial^2}{\partial q^{\nu}\, \partial q^{\mu}} Q^{\nu\mu}(q) P \tag{1.3}$$

for the conditional probability density $P(q \mid q_0, t)$ to observe q at time $t_0 + t$ if q_0 was realized at time t_0 can be solved (formally, and in some cases explicitly) by a functional integral

$$P(q \mid q_0, t) = \int D\mu \exp\left[-\frac{1}{\eta} \int_{q(-t) = q_0}^{q(0) = q} L(q(\tau), \dot{q}(\tau), \eta)\, d\tau \right] \tag{1.4}$$

with some suitably chosen Lagrangian $L(q, \dot{q}, \eta)$ and measure of integration $D\mu$. In the limit of weak noise, provided the inverse $Q_{\nu\mu}$ of $Q^{\nu\mu}$ exists, one has

$$L(q, \dot{q}, 0) = L_0(q, \dot{q}) = \tfrac{1}{2} Q_{\nu\mu}(\dot{q}^{\nu} - K^{\nu})(\dot{q}^{\mu} - K^{\mu}) \tag{1.5}$$

In the same limit, making the ansatz

$$P_{\infty}(q) \sim \exp\left[-\frac{1}{\eta} \phi(q) \right] \tag{1.6}$$

for the time-independent solution of Eq. (1.3), that equ: ion reduces to the Hamilton–Jacobi equation

$$H(q, \partial\phi/\partial q) = 0 \tag{1.7}$$

whose Hamiltonian $H(q, p)$ is related to L_0 by the usual Legendre transformation

$$\begin{aligned} p_{\nu} &= \partial L_0 / \partial \dot{q}^{\nu} \\ L_0(q, \dot{q}) &= \sum_{\nu} p_{\nu} \dot{q}^{\nu} - H(q, p) \end{aligned} \tag{1.8}$$

and takes the form

$$H(q, p) = \tfrac{1}{2} Q^{\nu\mu} p_\nu p_\mu + K^\nu p_\nu \tag{1.9}$$

The Hamiltonian exists regardless of whether $Q^{\nu\mu}$ has an inverse. It is assumed that P_∞ is unique and obtained from the time-dependent solutions (1.4) of (1.3) for $t \to \infty$. Then we must have[5,6]

$$\phi(q) = \inf_{(\mathscr{A}_i)} \left[\inf \int_{q(-\infty) \in \mathscr{A}_i}^{q(0)=q} L_0(q, \dot{q})\, d\tau + C(\mathscr{A}_i) \right] \tag{1.10}$$

where the infimum is taken over all paths connecting the attractor $\mathscr{A}_i$ of the deterministic dynamical system with the point q, and the infimum over all attractors is also taken. The constants $C(\mathscr{A}_i)$ are fixed by the condition of continuity of the expression $[\cdots]$ in the saddle on the separatrix separating two neighboring domains of attraction.

The function $\phi(q)$ has some nice properties in common with a thermodynamic potential, and in fact, it is reduced to such a potential if Eq. (1.1) describes fluctuations in thermodynamic equilibrium. Equation (1.6) may therefore be viewed as a generalized Boltzmann Einstein formula. The drift $K^\nu(q)$ which governs the deterministic dynamics satisfies a second law with respect to $\phi(q)$,

$$\frac{d\phi(q(t))}{dt} = K^\nu(q) \frac{\partial \phi(q)}{\partial q^\nu} = -\frac{1}{2} Q^{\nu\mu}(q) \frac{\partial \phi(q)}{\partial q^\nu} \frac{\partial \phi(q)}{\partial q^\mu} \leqslant 0 \tag{1.11}$$

because Eqs. (1.7), (1.9) imply that

$$\begin{gathered} K^\nu(q) = -\frac{1}{2} Q^{\nu\mu}(q) \frac{\partial \phi(q)}{\partial q^\mu} + r^\nu(q) \\ r^\nu(q) \frac{\partial \phi(q)}{\partial q^\nu} = 0 \end{gathered} \tag{1.12}$$

which, in turn, implies Eq. (1.11). Due to Eq. (1.11) the potential $\phi(q)$ may serve as a Lyapunov function for the deterministic dynamics. Usually, the properties (1.11), (1.12) are considered to be special properties of systems in thermodynamic equilibrium—which is really not true, as shown by the generality in which they are obtained. Special to systems in thermodynamic equilibrium is only the time-reversal property of $r^\nu(q)$, which then transforms like $\dot{q}^\nu$, and of $(-1/2)\, Q^{\nu\mu}\, \partial\phi/\partial q^\mu$, which then transforms like q^ν. These special symmetries are absent in the general case, which makes the determination of $\phi(q)$ a nontrivial task. The lack of simple time-reversal symmetries in Eq. (1.12), in the general case, has important consequences for the analytical properties of $\phi(q)$.[4] While the $K^\nu(q)$ are usually given as smooth, continuously differentiable functions, the two pieces on the right-hand side of Eq. (1.12) in general do not share this property. In fact, $\phi(q)$ as determined from Eq. (1.10) is continuous but need not be differentiable, leading to discontinuities in $r^\nu(q)$. In thermodynamic equilibrium, on the other hand, $r^\nu(q)$ is given by the reversible part of $K^\nu(q)$, from which it then inherits its continuous differentiability.

Two methods for finding $\phi(q)$ are, in principle, available in the general case, and will be exemplified below: (1) the solution of the Hamilton–Jacobi equation (1.7) which one must construct, in nontrivial cases, as a power series in some suitably chosen small parameter, or (2) a direct evaluation of the integral formula (1.10). The first method is, of course, not always applicable. In particular, it breaks down in regions of configuration space where $\phi(q)$ is not differentiable. This method also suffers from the difficulty that the Hamilton–Jacobi equation is only a local statement about $\phi(q)$ and it is often necessary to use Eq. (1.10) in addition, in order to single out the globally relevant solution. The second method is, in principle, always applicable, but usually it has to be implemented numerically and does not lead to analytical solutions.

In the following we consider two applications to systems undergoing bifurcations.

2. A CODIMENSION-2 BIFURCATION

A general discussion of codimension-2 bifurcations is given in Guckenheimer and Holmes.[7] As a physical example, convection of a binary fluid in a porous medium has recently been studied both theoretically[8] and experimentally.[9] A number of theoretical studies concerning the influence of noise on codimension-2 bifurcation have also recently appeared.[10–12] Let us consider as an example a codimension-2 bifurcation governed by the normal form[7]

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \mu_1 x + \mu_2 v + x^3 - x^2 v + Q^{1/2}\xi(t) \end{aligned} \tag{2.1}$$

where $\xi(t)$ is Gaussian white noise with intensity η. The codimension-2 point is reached for $\mu_1 \to 0$, $\mu_2 \to 0$ and we wish to obtain an expression for $\phi(x, v)$ which is accurate in that limit. It is therefore useful to rescale[7,11]

$$\begin{aligned} &\mu_{1,2} \to \varepsilon^2 \mu_{1,2} \\ &x \to \varepsilon x, \qquad v \to \varepsilon^2 v, \qquad t \to t/\varepsilon \\ &Q \to \varepsilon^5 Q \end{aligned} \tag{2.2}$$

and to construct $\phi(x, v)$ as a power series in ε. This strategy works[11] because, after the rescaling (2.2), the system (2.1) in the limit $\varepsilon \to 0$ becomes a conservative system[7] weakly perturbed by dissipation and noise. The potential $\phi(x, v)$ to leading order $[O(\varepsilon)]$ in ε is then only a function of the nearly conserved energy

$$E = \frac{v^2}{2} - \frac{\mu_1}{2} x^2 - \frac{1}{4} x^4 \tag{2.3}$$

which is obtained[11] in terms of the quadratures

$$\phi(E(x, v)) = \frac{2}{Q} \int_0^E d\tilde{E}\, [\bar{x}^2(\tilde{E}) - \mu_2] \tag{2.4}$$

$$\bar{x}^2(E) = \frac{\int_{x_1(E)}^{x_2(E)} x^2 v(E, x)\, dx}{\int_{x_1(E)}^{x_2(E)} v(E, x)\, dx} \tag{2.5}$$

where

$$v(E, x) = +(2E + \mu_1 x^2 + \tfrac{1}{2} x^4)^{1/2} \tag{2.6}$$

and $x_1(E)$, $x_2(E)$ are defined by

$$v(E, x_{1,2}) = 0; \qquad x_1 \leqslant x_2 \tag{2.7}$$

Corrections to ϕ of higher order in ε may be calculated, if desired. For a calculation of higher order terms in the case of a Hopf bifurcation see ref. 13.

It is clear from (2.3)–(2.4) that a local minimum of ϕ [i.e., an attractor of (2.1)] exists only for $\mu_1 < 0$ and even then the system is unstable for sufficiently large values of x or v, in fact, for $E(x, v) > V_0 = \mu_1^2/4$. The relevant domain of E is therefore $0 \leqslant E < V_0$. The local minimum of ϕ is at $E = 0$, i.e., at $x = v = 0$, for $\mu_2 < 0$, where $(\partial\phi/\partial E)_{E=0} > 0$. For $\mu_2 > 0$ the local minimum of ϕ, given by Eq. (2.4), moves to a finite value of E, $E = E_c$, given by $\mu_2 = \bar{x}^2(E_c)$. The attractor is then the limit cycle $v^2 + |\mu_1|\, x^2 - \frac{1}{2} x^4 = 2E_c$. The value of E_c lies in the relevant domain $0 \leqslant E_c < V_0$ if $\mu_2 \geqslant 0$, and $\mu_2 < -\mu_1/5$. For $\mu_2 = -\mu_1/5$ and $\mu_1 < 0$ one has $E = E_c$ and the limit cycle disappears by a heteroclinic bifurcation, where it forms a heteroclinic connection of the two saddle points $x = \pm(-\mu_1)^{1/2}$, $v = 0$ of the energy (2.3). In this way the "landscape" formed by the potential gives a clear and intuitive picture of all details of those bifurcations near $\mu_1 = \mu_2 = 0$ for which local attractors exist, which are just the cases of physical interest. But in addition ϕ also contains the relevant information on the effects of weak noise, as is clear from Eq. (1.6). Thus it can be used to calculate the mean exit times from the domains of the attractors at the origin or at the limit cycle.[14] The noise can be considered weak as long as $\Delta\phi \gg \eta$, where $\Delta\phi$ is the potential barrier the system has to climb from the attractor to the separatrix at $E = V_0$. Close to the bifurcation lines at $\mu_1 = 0$ and $\mu_2 = -\frac{1}{5}\mu_1$ two scaling regimes for the mean exit time τ are predicted by the theory.[2,14]

(i) $\mu_2 < 0$, $\mu_1 \to -0$:

$$|\mu_2| \tau \sim X^{-1} \exp X \tag{2.8}$$

where the scaling variable X is given by

$$X = \frac{|\mu_2| \mu_1^2}{2\eta Q}$$

(ii) $\mu_1 < 0$, $\mu_2 - \frac{1}{5}|\mu_1| \to -0$:

$$|\mu_1|^{1/2} \tau \sim Y^{-1/2} \exp Y \tag{2.9}$$

where the scaling variable Y is given by

$$Y = \frac{|\mu_1| (\mu_2 - |\mu_1|/5)^2}{\eta Q |\ln(|\mu_2 - |\mu_1|/5|/|\mu_1|)|}$$

The conditions of weak noise in the two regimes are $X \gg 1$, $Y \gg 1$, respectively.

3. NOISE IN A PENDULUM UNDER AN EXTERNAL TORQUE

As an example for the direct evaluation of the integral expression (1.10) we consider the stochastic dynamical system

$$\frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} - f(x) + F + (2\gamma)^{1/2} \xi(t) \tag{3.1}$$

with the parameters γ, F, and the Gaussian white noise $\xi(t)$ of intensity η, which not only describes a driven stochastic pendulum, but also, e.g., noise in a current-driven Josephson junction. The function $f(x)$ is 2π-periodic in x, e.g., $f(x) = \sin x$. For a fixed value of γ and increasing the value of $|F|$ starting from 0, the deterministic system corresponding to (3.1) with $\eta = 0$ passes through a regime with a single attracting fixed point for $|F| < F_c(\gamma)$, and then a coexistence regime of an attracting fixed point and a limit cycle for $F_c(\gamma) < |F| < 1$, and finally a regime with a single attracting limit cycle for $|F| > 1$.

The integral formula (1.10) in the present case reads

$$\phi(x, v) = \inf_{(i=0,1)} \phi_i(x, v)$$

$$\phi_i(x, v) = \inf \int_{(x(-\infty), \dot{x}(-\infty)) \in \mathscr{A}_i}^{(x(0), \dot{x}(0)) = (x, v)} dt \times \frac{1}{4\gamma} [\ddot{x} + \gamma \dot{x} + f(x) - F]^2 + C(\mathscr{A}_i) \tag{3.2}$$

where the $C(\mathscr{A}_i)$ are determined (up to a common additive constant) by the condition of continuity of ϕ in the saddle point on the separatrix separating the two coexisting domains of attraction for $F_c(\gamma) < |F| < 1$. Here and in the following the fixed-point attractor is denoted by $\mathscr{A}_0$, the limit cycle by $\mathscr{A}_1$. The minimizing trajectories in Eq. (3.2) satisfy an Euler–Lagrange equation which is of fourth order in time. Fortunately, there exist two exact integrals of motion,[6] the "energy" (1.9), which must vanish, and the less obvious time-dependent integral

$$A \equiv \left[\frac{2\gamma \dot{x}}{\ddot{x} + f(x) - F + \gamma \dot{x}} - 1 \right] e^{\gamma t} \tag{3.3}$$

which only exists in the case where the "energy" vanishes. Using the integral (3.3) in (3.2), we are left with

$$\phi_i(x, v) = \int_{(x(-\infty), \dot{x}(-\infty)) \in \mathscr{A}_i}^{(x(0), \dot{x}(0)) = (x, v)} d\tau \, \frac{\gamma \dot{x}^2}{(1 + Ae^{-\gamma\tau})^2} + C(\mathscr{A}_i) \tag{3.4}$$

where the integral is taken along a curve satisfying Eq. (3.3) and the required initial and final conditions. The latter are compatible, by construction, with the zero-energy condition, but the value of A has to be adjusted for (3.3) to be compatible with the boundary conditions in (3.4). This is best done by reversing time, so that a trajectory must satisfy

$$\ddot{x} = \frac{A - e^{-\gamma t}}{A + e^{-\gamma t}} \gamma \dot{x} - f(x) + F \tag{3.5}$$

with the initial and final conditions

$$(x(0), -\dot{x}(0)) = (x, v); \qquad (x(\infty), -\dot{x}(\infty)) \in \mathscr{A}_i \tag{3.6}$$

It is clear from (3.5), (3.6) that $A = 0$ for $i = 0$ and all points $(x, -v)$ in the domain of attraction of the fixed point $\mathscr{A}_0$. The curious reversal of the sign of v has its origin in the time reversal which led to (3.5). Evaluating Eqs. (3.4)–(3.6) for this case, one finds

$$\phi_0(x, v) = \tfrac{1}{2}v^2 + g(x) - Fx + C_0 \tag{3.7}$$

where $f(x) = g'(x)$ and the constant C_0 includes $C(\mathscr{A}_0)$ of Eq. (3.4). In general the constant A can be determined numerically from Eqs. (3.5) and (3.6), and then the integral (3.4) can also be evaluated numerically. In essence, this method was used in ref. 6 to compute the potential $\phi_1(x, v)$. Actually, a slightly different and more convenient version of this procedure was employed, in which, for given initial x, v, the value of the quantity $p_v(x, v) = v/[A(x, v) + 1]$ was determined [which is, of course, equivalent to determining $A(x, v)$] such that the backward integrated dynamics (3.3) landed on the attractor $\mathscr{A}_1$. The meaning of p_v is that of the canonically conjugate momentum associated with v in the Hamiltonian formulation of the weak-noise limit given in Section 1. Its particular choice, according to the prescription given, means placing the initial point on the unstable manifold which the attractor $\mathscr{A}_1$ develops when it is embedded in the phase space of the Hamiltonian dynamics.[4,6] The advantage of the use of $p_v(x, v)$ instead of $A(x, v)$ consists in the fact that $p_v(x, v)$ is related to $\phi_1(x, v)$ simply by[4,6]

$$\partial\phi_1(x, v)/\partial v = p_v(x, v) \tag{3.8}$$

i.e., the integral (3.4) is reduced to

$$\phi_1(x, v) = \int_{v_1(x)}^{v} p_v(x, \tilde{v})\, d\tilde{v} + C_1 \tag{3.9}$$

where $v = v_1(x)$ is the equation of the limit cycle $\mathscr{A}_1$. Equations (3.8), (3.9) are the familiar relations between the action and the canonical momentum. For a detailed discussion of the results of this method and the problem of taking the infimum among the potentials ϕ_0, ϕ_1 in the coexistence region of $\mathscr{A}_0$ and $\mathscr{A}_1$ we refer to ref. 6. Alternative methods for the computation of the barrier of ϕ from the limit cycle to the saddle have been described in refs. 15 and 16.

In summary, the weak-noise limit of macroscopic autonomous nonequilibrium systems has much in common with equilibrium thermodynamics, the essential difference being the existence of simple transformation laws under time reversal for macroscopically defined quantities in thermodynamic equilibrium. The two available methods for the construction of a generalized thermodynamic potential—a power series expansion in a small parameter and evaluation of a minimum principle—have been exemplified for a case of codimension-2 bifurcation and Brownian motion of a pendulum under an external torque.

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